MATH 5061 Lecture on 2/26/2020

Last time

On any C^∞ mfd M^m, defined two differential operators:

- Exterior derivative $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

locally $d(\alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = d\alpha_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

- well-defined since $d \circ F^* = F^* \circ d$

Key Properties: $d^2 = 0$; $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta$

where $\omega \in \Omega^r$, $\eta \in \Omega^s$

- Lie derivative (w.r.t. $X \in \mathfrak{X}(M)$) $\mathcal{L}_X : \mathcal{T}(T^r_s M) \longrightarrow \mathcal{T}(T^r_s M)$

$X \rightsquigarrow$ flow $\{\varphi_t\} \subseteq \mathrm{Diff}(M)$ $\qquad \mathcal{L}_X(\alpha) := \frac{d}{dt}\Big|_{t=0} \varphi_t^{\#} \alpha$

$X = \frac{d}{dt}\Big|_{t=0} \varphi_t$

Key Properties: $\mathcal{L}_X Y = [X, Y]$; $\mathcal{L}_X \circ c = c \circ \mathcal{L}_X$

because of this $\rightarrow$ $\mathcal{L}_X(\omega \otimes \eta) = (\mathcal{L}_X \omega) \otimes \eta + \omega \otimes (\mathcal{L}_X \eta)$

Note: $\mathcal{L}_X : \Omega^k(M) \longrightarrow \Omega^k(M)$

Q: How to relate d and $\mathcal{L}_X$ on forms?

$$\begin{tikzcd}
\Omega^0(M) \arrow[loop above, "\mathcal{L}_X"] \arrow[r, shift left, "d"] \arrow[d, phantom, "\parallel"] & \Omega^1(M) \arrow[loop above, "\mathcal{L}_X"] \arrow[l, shift left, "\iota_X"] \arrow[r, shift left, "d"] & \Omega^2(M) \arrow[loop above, "\mathcal{L}_X"] \arrow[l, shift left, "\iota_X"] \arrow[r, shift left, "d"] & \cdots \arrow[l, shift left, "\iota_X"] \arrow[r, shift left, "d"] & \Omega^k(M) \arrow[loop above, "\mathcal{L}_X"] \arrow[l, shift left, "\iota_X"] \arrow[r, shift left, "d"] & \cdots \arrow[l, shift left, "\iota_X"] \arrow[r, shift left, "d"] & \Omega^m(M) \arrow[loop above, "\mathcal{L}_X"] \arrow[l, shift left, "\iota_X"] \\
C^\infty(M) & & & & \text{Cartan: } \mathcal{L}_X = \iota_X \circ d + d \circ \iota_X \arrow[u] & &
\end{tikzcd}$$

Lemma: $\boxed{d \circ \mathcal{L}_X = \mathcal{L}_X \circ d}$

Pf: Use linearity.

$$\underset{\substack{\parallel \\ d \circ \mathcal{L}_X(\omega)}}{d\left(\frac{\varphi_t^* \omega - \omega}{t}\right)} = \frac{d(\varphi_t^* \omega) - d\omega}{t} = \underset{\substack{\parallel \\ \mathcal{L}_X(d\omega)}}{\frac{\varphi_t^*(d\omega) - d\omega}{t}}$$

as $t \to 0$, $\qquad d \circ \mathcal{L}_X(\omega)$ $\qquad \mathcal{L}_X(d\omega)$. $\qquad \square$

Interior Product (w.r.t. $X \in \mathfrak{X}(M)$)

$\exists$ operator $\iota_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ s.t.

$$\iota_X(\omega)\big(Y_1, \dots, Y_{k-1}\big) := \omega(X, Y_1, \dots, Y_{k-1})$$

Key Property: $\iota_X(\omega \wedge \eta) = (\iota_X \omega) \wedge \eta + (-1)^r \omega \wedge (\iota_X \eta)$

where $\omega \in \Omega^r$, $\eta \in \Omega^s$.

"Pf": On 2-forms,

$$\iota_X(\underset{\uparrow}{\omega_1} \wedge \underset{\nearrow}{\omega_2})\big(Y\big) := (\omega_1 \wedge \omega_2)(X, Y)$$

1-forms

$$= \det \begin{pmatrix} \omega_1(X) & \omega_2(X) \\ \omega_1(Y) & \omega_2(Y) \end{pmatrix}$$

$$= (\omega_1(X)\,\omega_2)(Y) - (\omega_2(X)\,\omega_1)(Y)$$

$$= \big(\iota_X(\omega_1) \wedge \omega_2 - \omega_1 \wedge (\iota_X \omega_2)\big)(Y).$$

$\square$

Cartan's Majic Formula: $\boxed{\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X}$ on $\Omega^k(M)$.

- On functions ($\Omega^0(M)$), let $f \in C^\infty(M)$.

$$\mathcal{L}_X f = X(f)$$

$$\|$$

$$(\iota_X \circ d + d \circ \iota_X)(f) = \iota_X(df) + 0 = df(X) = X(f).$$

- On 1-forms $\omega \in \Omega^1(M)$,

$$(\mathcal{L}_X \omega)(Y) = X(\omega(Y)) - \omega([X, Y])$$

$\|$ Cartan

$$(\iota_X \circ d + d \circ \iota_X)(\omega)\big(Y\big) = \iota_X(d\omega)(Y) + d\big(\overbrace{\iota_X \omega}^{C^\infty(M)}\big)(Y)$$

$$= d\omega(X, Y) + Y\big(\omega(X)\big)$$

$$\Rightarrow \quad \boxed{d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])}$$

"invariant" formula for d on $\Omega^1(M)$

Proof of Cartan's Formula: Define $L'_X = \iota_X \circ d + d \circ \iota_X : \Omega^k \to \Omega^k$.

GOAL: $L_X = L'_X$ on Ω^k for all k.

Step 1: $L_X = L'_X$ on $\Omega^0(M) = C^\infty(M)$.

Step 2: $L'_X \circ d = d \circ L'_X$ on Ω^k, $\forall k$.

Pf:

$$L'_X \circ d = (\iota_X \circ d + d \circ \iota_X) \circ d$$

$$= \iota_X \circ \cancel{d^2}^{\,0} + d \circ \iota_X \circ d$$

same

$$d \circ L'_X = d \circ (\iota_X \circ d + d \circ \iota_X)$$

$$= d \circ \iota_X \circ d + \cancel{d^2}^{\,0} \circ \iota_X$$

Fact: $d^2 = 0$

Step 3: $L'_X(\alpha \wedge \beta) = (L'_X \alpha) \wedge \beta + \alpha \wedge (L'_X \beta)$ $\quad \forall \alpha \in \Omega^r, \forall \beta \in \Omega^s$

Pf: $L'_X(\alpha \wedge \beta)$

$$= \iota_X \circ d(\alpha \wedge \beta) + d \circ \iota_X(\alpha \wedge \beta)$$

$$= \iota_X [d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta]$$

$$+ d[(\iota_X \alpha) \wedge \beta + (-1)^r \alpha \wedge (\iota_X \beta)]$$

$$= \boxed{\begin{aligned} &\iota_X(d\alpha) \wedge \beta \\ &+ d(\iota_X \alpha) \wedge \beta \end{aligned}} \begin{aligned} &+ (-1)^{r+1} d\alpha \wedge (\iota_X \beta) + (-1)^r d\alpha \wedge (\iota_X \beta) + \\ &+ (-1)^{r-1} (\iota_X \alpha) \wedge d\beta + (-1)^r d\alpha \wedge (\iota_X \beta) + \end{aligned} \boxed{\begin{aligned} &(-1)^{2r} \alpha \wedge \iota_X d\beta \\ &(-1)^{2r} \alpha \wedge d\iota_X \beta \end{aligned}}$$

cancel cancel

$$= L'_X \alpha \wedge \beta + \alpha \wedge L'_X \beta$$

Step 4: $L_X = L'_X$ on $\Omega^1(M)$.

Suffices to check $L_X \omega = L'_X \omega$ where $\omega = u\,dv \in \Omega^1(M)$

$$L'_X(u\,dv) \underset{\text{Step 3}}{=} (L'_X u) \cdot dv + u \cdot L'_X(dv) \overset{\text{Step 1 \& 2}}{=} X(u)\,dv + u\,d(L'_X v)$$

$$\underset{\text{Step 1}}{=} X(u)\,dv + u\,d(X(v)) = L_X(u\,dv)$$

Step 3 induction

DONE!

$\blacksquare$

Applications of $\mathcal{L}_X$

(A) Structure-preserving symmetries

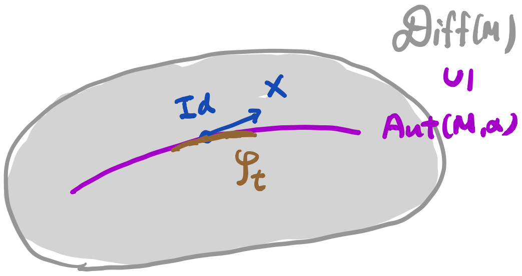
Philosophy: M^m equipped w/ additional "structures" defined by a tensor $\alpha \in \mathcal{T}(T^r_s M)$.

$$\mathrm{Aut}(M,\alpha) := \{ \varphi \in \mathcal{D}iff(M) \mid \varphi^{\#}\alpha = \alpha \} \subseteq \mathcal{D}iff(M)$$

Note: $\underset{\overset{\|}{\left.\frac{d}{dt}\right|_{t=0} \varphi_t^{\#}\alpha}}{\mathcal{L}_X \alpha \equiv 0} \overset{(Ex.)}{\Longleftrightarrow} \varphi_t^{\#}\alpha \equiv \alpha \ \forall t$ where $\{\varphi_t\} \subseteq \mathcal{D}iff(M)$ gen. by $X \in \mathfrak{X}(M)$

$\Longleftrightarrow \{\varphi_t\} \subseteq \mathrm{Aut}(M,\alpha)$

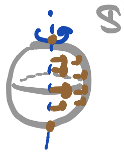
$$\left\{ \begin{matrix} \text{infinitesimal} \\ \text{symmetries } X \end{matrix} \right\} = \{ X \in \mathfrak{X}(M) \mid \mathcal{L}_X \alpha = 0 \}$$



1] (M^n, g) Riemannian mfd ; g : $(0,2)$-tensor, pos. def. symmetric

$$\mathrm{Aut}(M,g) = \mathrm{Isom}(M,g) = \{ \underbrace{\varphi : M \xrightarrow{\text{diffeo.}} M, \ \varphi^{*}g = g}_{\text{isometries}} \}$$

E.g.) $\mathrm{Isom}(S^n, g_{round}) = O(n+1)$.



infinitesimal isometry X : $\boxed{\mathcal{L}_X g = 0}$ Killing field

2] (M^{2n}, ω) symplectic mfd ; $\underbrace{\omega \in \Omega^2(M) \text{ non-deg., closed}}_{\text{symplectic form}}$ (i.e. $d\omega = 0$)

$$\mathrm{Aut}(M,\omega) = \mathrm{Sym}(M,\omega) = \{ \text{symplectomorphisms} \} = \{ \varphi^{*}\omega = \omega \}$$

infinitesimally, $\mathcal{L}_X \omega = 0 \overset{\text{Cartan}}{\Longleftrightarrow} (\cancel{\iota_X \circ d}^{\,0} + d \circ \iota_X)\omega = 0$ $\because d\omega = 0$

$\Longleftrightarrow \boxed{\iota_X \omega \text{ is a closed 1-form.}}$

X : symplectic v.f.

3] (M^{2n}, J) almost complex mfd ; $J \in T(T^1_1 M)$ $\quad J : TM \to TM$

$J^2 = -id$ "almost cpx. str."

$$\mathrm{Aut}(M,J) = \mathrm{Hol}(M,J) = \{ J\text{-holo. } \varphi : M \to M \text{ st. } \underbrace{\varphi_* \circ J = J \circ \varphi_*}_{\text{Cauchy-Riemann eq}^{\underline{n}}} \}$$

infinitesimally, $\boxed{L_X J = 0}$ holo. v.f. X

(B) Frobenius Thm Revisited

Recall: L^r : r-dim distribution in M^m

locally, $L^r(p) = \mathrm{Span}\{ \underbrace{X_1(p), \dots, X_r(p)}_{l.i} \} \le T_pM$

Frobenius Thm: L is integrable $\Leftrightarrow$ $[X_i, X_j] \in L^r$ for $i,j = 1, \dots, r$

"F.C."

Q: dual formulation?

complete to basis: $\underbrace{X_1, \dots, X_r}_{\text{tangent to } L}$ $\quad X_{r+1}, \dots, X_m$ on TM

dual basis: $\omega^1, \dots, \omega^r$ $\quad \omega^{r+1}, \dots, \omega^m$ on T^*M.

$$\Rightarrow \quad L^r(p) = \mathrm{Span}\{ X_1(p), \dots, X_r(p) \}$$
$$= \{ v \in T_pM \mid \omega^\alpha(v) = 0, \ \alpha = r+1, \dots, m \}$$

Recall: $\omega^\alpha \in \Omega^1$, $\alpha = r+1, \dots, m$

$$d\omega^\alpha(X_i, X_j) = X_i(\overset{0}{\overset{\shortparallel}{\omega^\alpha(X_j)}}) - X_j(\overset{0}{\overset{\shortparallel}{\omega^\alpha(X_i)}}) - \underset{\underset{0}{\shortparallel}}{\omega^\alpha([X_i, X_j])} \quad \Leftrightarrow (F.C.)$$

$i,j = 1, \dots, r$

Thus, (F.C.) $\Leftrightarrow$ $d\omega^\alpha(X_i, X_j) = 0$ $\quad$ (F.C.)*

for $\alpha = r+1, \dots, m$

$i,j = 1, \dots, r$

Connections and Curvature (Chern Ch.4)

"std" derivative on $\mathbb{R}^3$

Recall: $S^2 \subseteq \mathbb{R}^3 \rightsquigarrow \nabla_X Y := (D_X Y)^T$ covariant derivative

$\rightsquigarrow$ parallelism, geodesics, Gauss-Codazzi eq$^{\underline{n}}$. etc......

Q: How to differentiate $\Gamma(E)$ of a vector bundle $\pi : E \to M$?

Def$^{\underline{n}}$ of connections: Let $\pi : E \to M$ be a C^∞ vector bundle.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r] & E \arrow[d, "\pi"] \\ & M \arrow[u, bend left, "s"] \end{tikzcd}$$

A "connection on E" is a map

$$D : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$

$$(X, s) \longmapsto D_X s$$

s.t. (1) linearity: $D_X(s_1 + s_2) = D_X s_1 + D_X s_2$

$D_{X+Y} s = D_X s + D_Y s$

(2) tensorial in X: $D_{fX} s = f D_X s \leftarrow \forall f \in C^\infty(M)$

(3) Liebniz in s: $D_X(fs) = X(f)\, s + f D_X s$

Lemma: Fix $p \in M$. Then $D_X s\,(p)$ depends on $X(p)$, and s in a nbd. of p

Pf: Same as "characterization of tensors".

Alternatively, we can regard

$$\begin{array}{ccc} D : \Gamma(E) & \longrightarrow & \Gamma(T^*M \otimes E) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s & \longmapsto & Ds(X) := D_X s \end{array}$$

s.t. $\begin{cases} D(s_1 + s_2) = Ds_1 + Ds_2 \\ D(fs) = df \otimes s + f Ds \end{cases}$

Goal: Recognize D locally as matrices of 1-forms.

Linear Algebra Level: V = m-dim vector space / $\mathbb{R}$

- Let $e_1, \dots, e_m$ be a basis.

$$V \ni v = \sum_{i=1}^{m} x^i e_i = (e_1 \cdots e_m) \begin{pmatrix} x^1 \\ \vdots \\ x^m \end{pmatrix} = \underline{\underline{e}}\, X$$

If $T: V \to V$ linear, then

$$T e_i = \sum_{j=1}^{m} a_i^j e_j \quad , \quad [T]_\beta = (a_i^j) =: A$$

i.e. $$T\underline{\underline{e}} = \underline{\underline{e}} A \quad \text{or} \quad T(\underline{\underline{e}} X) = \underline{\underline{e}} A X$$

Connection matrix for $D: \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$

- let $s_1, \dots, s_r$ be locally defined frame fields on E.

$$\Gamma(\underline{T^*M} \otimes E) \ni D s_\alpha = \sum_{\beta=1}^{r} \underline{\omega_\alpha^\beta} \otimes s_\beta \quad \text{where } \omega_\alpha^\beta \text{ locally 1-forms}$$

i.e. Fix $X \in \mathfrak{X}(M)$, $$D_X s_\alpha = \sum_\beta \omega_\alpha^\beta(X)\, s_\beta$$

Idea: $$D \overset{s_1,\dots,s_r}{\overset{\curvearrowright}{=}} (\omega_\alpha^\beta)_{\alpha,\beta=1,\dots,r}$$ locally $r \times r$ matrix of 1-forms

Notation: $\underline{\underline{s}} = (s_1, \dots, s_r)$, $\omega = (\omega_\alpha^\beta)$ $\Rightarrow$ $$\boxed{D\underline{\underline{s}} = \underline{\underline{s}}\,\omega}$$

Example (Standard connection on $\mathbb{R}^m$)

$\mathbb{R}^m$ $\quad x^1, \dots, x^m$ std coord. $\quad E = T\mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^m$

(1) $\underbrace{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}}_{\text{frame field for } T\mathbb{R}^m}$ global, parallel $\Rightarrow$ $D\left(\frac{\partial}{\partial x^i}\right) \equiv 0$ $\quad (\Leftrightarrow \omega \equiv 0)$

(2) In general, consider a local frame field $s_1, \dots, s_m$.

$$s_\alpha = \sum_\beta a_\alpha^\beta \frac{\partial}{\partial x^\beta} \rightsquigarrow A = (s_1 \cdots s_m) = \begin{pmatrix} a_1^1 & \cdots & a_m^1 \\ \vdots & & \vdots \\ a_1^m & \cdots & a_m^m \end{pmatrix}$$

invertible $m \times m$ matrix

Let $\underline{\underline{s}} = (s_1 \cdots s_m)$.

$$D \underline{\underline{s}} = \underline{\underline{s}}\, \omega$$ where ω = connection matrix of 1-forms w.r.t. $\underline{\underline{s}}$

$$\Longleftrightarrow \quad dA = A\omega \quad (*)$$

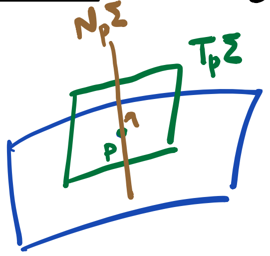
$\because$ A invertible $\Rightarrow$ $\boxed{\omega = A^{-1} dA}$

Differentiate $(*)$.

$$0 = d^2 A = d(A\omega) = dA \wedge \omega + A\, d\omega$$
$$\overset{(*)}{=} (A\omega) \wedge \omega + A\, d\omega$$
$$= A(\omega \wedge \omega + d\omega)$$

i.e. $\underbrace{\boxed{d\omega + \omega \wedge \omega = 0}}_{\text{"curvature"}}$ as matrix of Ω^2.

<u>Example</u>: Moving frames on submfd $\Sigma^k \subseteq \mathbb{R}^m$.



$\Sigma^k \subseteq \mathbb{R}^m$ D: std connection.

<u>Tangent bundle</u>:
$$\begin{tikzcd} T\Sigma \arrow[d, "\pi"] \\ \Sigma \end{tikzcd}$$

$D_X^T Y := (D_X Y)^T$

D^T: induced connection

$D_X Y = (D_X Y)^T + (D_X Y)^N$

$X, Y \in \mathfrak{X}(\Sigma)$

<u>Normal bundle</u>:
$$\begin{tikzcd} N\Sigma \arrow[d, "\pi"] \\ \Sigma \arrow[u, bend left, "s"] \end{tikzcd}$$

D^N: induced connection

$D_X^N s := (D_X s)^N$

Locally, we find a "moving frame":

$e_1, \dots e_k$ basis $T\Sigma$

$e_{k+1}, \dots, e_m$ basis $N\Sigma$

$A = (e_1 \cdots e_m)$ basis of $\mathbb{R}^m$

As before,

$$\omega = A^{-1}dA = \begin{array}{cc} \begin{array}{cc} \overbrace{\qquad}^{k} & \overbrace{\qquad}^{m-k} \end{array} & \\ \left(\begin{array}{c|c} \omega^T & \omega^i{}_\alpha \\ \hline \omega^\alpha{}_i & \omega^N \end{array}\right) & \begin{array}{l} \}\,k \\ \}\,m-k \end{array} \end{array}$$

flat connection of D w.r.t. A

$i = 1, \dots, k$

$\alpha = k+1, \dots, m$

$$h(e_i, e_j) := (D_{e_j} e_i)^N = \sum_{\alpha=k+1}^{m} \omega^\alpha{}_i(e_j)\, e_\alpha$$

2nd f.f.

Remember, $\mathbb{R}^m$ is flat,

$$0 \equiv d\omega + \omega \wedge \omega = \left(\begin{array}{c|c} \text{Gauss} & \text{Codazzi} \\ \hline \text{Codazzi} & \text{Ricci} \end{array}\right)$$

i.e. $0 = d\omega^i{}_j + \sum_{a=1}^{m} \omega^i{}_a \wedge \omega^a{}_j$

$= d\omega^i{}_j + \sum_{l=1}^{k} \omega^i{}_l \wedge \omega^l{}_j + \sum_{\alpha=k+1}^{m} \omega^i{}_\alpha \wedge \omega^\alpha{}_j$

Gauss: $d\omega^T + \omega^T \wedge \omega^T$ = quad. terms of 2nd f.f.

Ricci: $d\omega^N + \omega^N \wedge \omega^N$ = quad. terms of 2nd f.f.

[**Fact:** when $m = k+1$, Ricci auto. true.]

Codazzi: a bit different involves derivatives of 2nd f.f.